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Concerning Linear Projective Order.

BY ARTHUR RICHARD SCHWEITZER.

INTRODUCTION.

Hölder, in the *Mathematische Annalen*, Vol. LXV (1908), p. 161, considers the geometry of the projective line. In particular, in §§ 1–5 he has given several treatments of the linear projective order theory. In our investigations in the logic of geometry we have been impressed with the desirability* of connecting the foundations of linear projective order as intimately as possible with the descriptive† line; and it would seem that the previous treatments‡ of the subject, including Hölder's, have not exhibited the close relation between the descriptive and projective linear orders as coherently as a suitable analysis of the theories in question really permits.§

In the present paper we aim to give an axiomatic analysis of linear projective order which has a very simple relation with a linear descriptive system.|| Aside from this advantage, a significant property of our analysis is the relation which it bears to our planar descriptive system 2R_2 ¶ and from which springs a new application of a fundamental logical principle. The latter may be stated** as follows :

*See concluding paragraphs of § 2 of the present paper and our article, "A Theory of Geometrical Relations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), pp. 365–410. In the sequel reference is made by pages to this paper by the notation "T. G. R."

†On the meaning of the term "descriptive" see B. Russell, "The Principles of Mathematics," Cambridge (1903), § 374; in T. G. R. we have used this term in a generalized sense.

‡For the literature reference may be made to Russell, *loc. cit.*, §§ 204–205, and to Hölder's article.

§Hölder, *loc. cit.*, p. 179, note 1, and p. 181, notes 2, 3.

||The system which is here relevant is our system 1R_1 , T. G. R., p. 378.

¶T. G. R., p. 382 and § 3 of this article.

**This statement does not differ essentially from that given by E. H. Moore, *Bulletin of the American Mathematical Society*, Vol. XVI (1909) pp. 111–112. The principle is essentially heuristic and accordingly may be compared with the empirical position of J. S. Mill, "A System of Logic," 4th edition, London (1856). In our specific application, by *concept* we understand *theory*.

The existence of resemblances between given concepts implies the existence of a general concept which underlies the particular concepts and unifies them with respect to those resemblances.

Like the system 2R_2 , our axioms are phrased in terms of an undefined relation $\alpha R \beta \gamma$ which may be interpreted concretely by the statement: "If a person swims from the point β to the point γ , then the point α is at his right."

§ 1. *A New Analysis of Linear Projective Order: The System ${}^2R_2^{(0)}$.*

I. AXIOMS.

1. There exists at least one point.*
2. The existence of α implies the existence of $\alpha_0, \beta_0, \gamma_0$ such that $\alpha_0 R \beta_0 \gamma_0$ or $\alpha_0 R \gamma_0 \beta_0$.
3. $\alpha R \beta \gamma$ implies $\alpha \bar{R} \gamma \beta$.†
4. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$.
5. $\alpha R \beta \gamma, \xi \neq \alpha, \beta, \gamma$ imply $\xi R \beta \gamma$ or $\alpha R \xi \gamma$.
6. $\alpha R \beta \gamma, \xi R \beta \gamma, \xi \neq \alpha$ imply $\alpha R \xi \gamma$ or $\xi R \alpha \gamma$.
7. $\alpha R \beta \gamma, \alpha \neq \beta, \beta \neq \gamma, \alpha \neq \gamma$ imply the existence of ξ such that $\beta R \xi \gamma$ and $\alpha R \xi \gamma$.

II. DEFINITIONS.

1. $\gamma \delta$ is *collinear* with $\alpha \beta$ ($\alpha, \beta \neq \gamma, \delta$) means, $\alpha R \beta \gamma$ or $\beta R \alpha \gamma$; $\alpha R \beta \gamma$ implies $\delta R \beta \gamma$ or $\alpha R \delta \gamma$; and $\beta R \alpha \gamma$ implies $\delta R \alpha \gamma$ or $\beta R \delta \gamma$ (compare Fig. 1; we have $\alpha R \beta \gamma$ and $\delta = \delta'_1$ or δ'_2 or δ'').

2. From definition 1 is obtained the definition of " $\gamma \delta$ separates $\alpha \beta$ " by the proper substitution of "and" for "or"; compare Fig. 2.

3. ξ is in the *interior* of $\alpha \beta$ means, $\alpha \neq \beta$ and the existence of γ such that $\alpha R \beta \gamma$ implies $\xi R \beta \gamma$ and $\alpha R \xi \gamma$; compare Fig. 3.‡

Instead of phrasing the preceding system of axioms (absolutely) in terms of an alternating relation R , one may construct an equivalent system

* Greek letters are used to denote points.

† The rule over the R is a symbol of negation.

‡ Compare the definitions, T. G. B., pp. 382–383.

(relatively) in terms of a transitive and symmetrical* relation K between triads† as follows:

System ${}^2K_2^{(0)}$.

1. There exists at least one point.
2. The existence of α implies the existence of $\alpha_0, \beta_0, \gamma_0$ such that $\alpha_0\beta_0\gamma_0 K$.
3. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma\bar{K}\alpha\gamma\beta$.
- 3'. $\alpha\beta\gamma K$ implies $\alpha\gamma\beta K$.
4. $\alpha\beta\gamma K$ implies $\alpha\beta\gamma K\beta\gamma\alpha$.
5. $\alpha\beta\gamma K, \xi \neq \alpha, \beta, \gamma$ imply $\xi\beta\gamma K$ or $\alpha\xi\gamma K$.

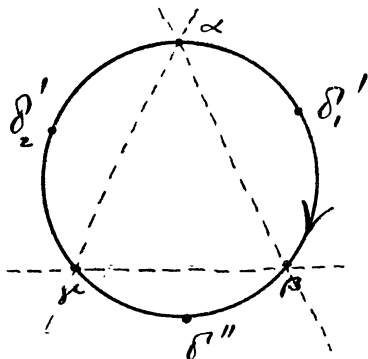


FIG. 1.

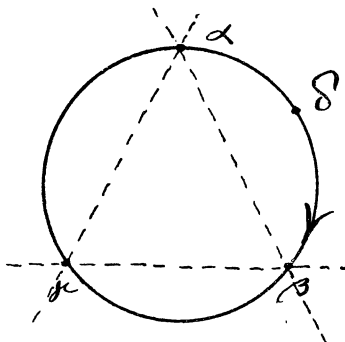


FIG. 2.

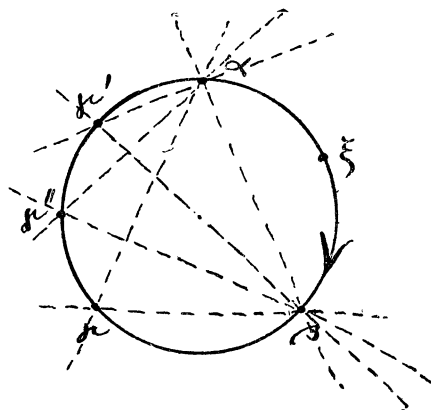


FIG. 3.

6. $\alpha\beta\gamma K, \xi\beta\gamma K, \xi \neq \alpha$ imply $\alpha\xi\gamma K$.
- 6'. $\xi\beta\gamma K\alpha\xi\gamma$ implies $\xi\beta\gamma K\alpha\beta\gamma$.
7. $\alpha\beta\gamma K, \alpha \neq \beta, \beta \neq \gamma, \alpha \neq \gamma$ imply the existence of ξ such that $\alpha\beta\gamma K\xi\gamma$ and $\alpha\beta\gamma K\alpha\xi\gamma$.
8. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha\beta\gamma K$.
9. $\alpha\beta\gamma K\alpha'\beta'\gamma'$ implies $\alpha'\beta'\gamma' K$.
10. $\alpha\beta\gamma K\xi\eta\zeta, \xi\eta\zeta K\alpha'\beta'\gamma'$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$.
11. $\alpha\beta\gamma K, \alpha'\beta'\gamma' K$ imply $\alpha\beta\gamma K\alpha'\beta'\gamma'$ or $\alpha\beta\gamma K\beta'\alpha'\gamma'$.

* For definitions see Russell, *loc. cit.*, § 208.

† Compare the abstract, *Bulletin of the American Mathematical Society*, Vol. XIII (1906) p. 58. On the notation of the system ${}^2K_2^{(0)}$ see T. G. R., pp. 375, 394. In connection with this system one may define "Gleichartigkeit" of triads as we have done in the case of tetrads, *Bull. Amer. Math. Soc.*, Vol. XV (1908), p. 80; compare Grassmann, *Gesammelte Werke*, Vol. I, Part I, p. 163; Hölder, *loc. cit.*, § 4, uses the same term in a different sense.

On the relation between the preceding systems ${}^2R_2^{(0)}$ and ${}^2K_2^{(0)}$ we may refer to the systems 2R_2 and 2K_2 .*

III. THEOREMS.

On the basis of the system ${}^2R_2^{(0)}$ one can easily prove :

1. $\alpha R\xi\gamma$ and $\xi R\beta\gamma$ imply $\alpha R\beta\gamma$.
2. $\alpha R\beta\gamma$ and $\xi R\beta\gamma$, $\xi \neq \alpha$ imply $\alpha R\xi\beta$ or $\xi R\alpha\beta$.
3. $\alpha R\beta\gamma$ and $\alpha R\xi\gamma$, $\xi \neq \beta$ imply $\xi R\beta\gamma$ or $\beta R\xi\gamma$.

Then with the aid of the preceding theorems it is easily shown that the axioms of Hölder, *loc. cit.*, § 5, follow from our axioms.

In contrast with the system 2R_2 it may also be shown that if $\alpha R\beta\gamma$ and $\xi \neq \alpha, \beta, \gamma$, then ξ is in the interior of precisely three compartments† associated with $\alpha\beta\gamma$; that is, one of the following three cases must hold :

I.	II.	III.
$\alpha R\beta\gamma$	$\alpha R\beta\gamma$	$\alpha R\beta\gamma$
$\alpha R\xi\gamma$	$\xi R\alpha\gamma$	$\alpha R\xi\gamma$
$\beta R\xi\gamma$	$\xi R\beta\gamma$	$\xi R\beta\gamma$
$\alpha R\beta\xi$	$\alpha R\beta\xi$	$\beta R\alpha\xi$

Thus the seven compartments into which a triangle separates a descriptive plane are here reduced, as it were, to three. Moreover, if $\alpha R\beta\gamma$, then $\xi R\beta\gamma$, $\alpha R\xi\gamma$ and $\alpha R\beta\xi$ together are impossible; likewise there is no point in the interior of both $\alpha\beta$ and $\beta\alpha$ ‡. There exists, however, a point ξ in the interior of $\alpha\beta$; and if $\alpha R\xi\gamma$ and $\xi R\beta\gamma$, then ξ is in the interior of $\alpha\beta$.

§ 2. *Relation to a Linear Descriptive System.*

The system ${}^2R_2^{(0)}$ is very simply related to the linear descriptive system 1R_1 .§ Namely, from the system 1R_1 can be derived an associated system by a process

* Cf. T. G. R., pp. 382, 393–394 and § 2 of the present paper. The latter is related exclusively to the system ${}^2R_2^{(0)}$ unless otherwise specified.

† The term *compartment* seems to be in closer harmony with the planar descriptive connotations of our paper than the term “Strecke” (segment) which Hölder uses.

‡ See T. G. R., p. 382, definitions 3, 4.

§ Cf. T. G. R., p. 378.

which may perhaps be called *tactical adjunction*; one writes throughout in 1R_1 , $\theta_1 R \theta_2 \gamma$ for $\theta_1 R \theta_2$. The following axioms are then obtained;*

1. There exists α .
2. The existence of α implies the existence of $\alpha_0, \beta_0 [\gamma]$ such that $\alpha_0 R \beta_0 \gamma$ or $\beta_0 R \alpha_0 \gamma$.
3. $\alpha R \beta \gamma$ implies $\beta \bar{R} \alpha \gamma$.
4. $\alpha R \beta \gamma$ and $\xi \neq \alpha, \beta$ imply $\alpha R \xi \gamma$ or $\xi R \beta \gamma$.
5. $\alpha R \beta \gamma$ and $\alpha R \xi \gamma, \xi \neq \beta$ imply $\xi R \beta \gamma$ or $\beta R \xi \gamma$.
6. $\alpha R \beta \gamma$ implies the existence of ξ_1 such that $\beta R \xi_1 \gamma$ and $\alpha R \xi_1 \gamma$.
7. $\alpha R \beta \gamma$ implies the existence of ξ_2 such that $\xi_2 R \alpha \gamma$ and $\xi_2 R \beta \gamma$.
8. $\alpha R \beta \gamma$ implies the existence of ξ such that $\alpha R \xi \gamma$ and $\xi R \beta \gamma$.

If now one assumes:

- 3'. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$

then axioms 7, 8 follow from 1–6, and the latter axioms become essentially the system ${}^2R_2^{(0)}$. This genesis of the system ${}^2R_2^{(0)}$ suggests at once that with reference to a fixed point \dagger of the system ${}^2R_2^{(0)}$ the set of axioms 1R_1 is definable. For we may define: \ddagger

$\delta R_{\alpha\beta\gamma}\varepsilon$ means: 1°. There exist α, β, γ such that $\alpha R \beta \gamma$. 2°. $\delta R \varepsilon \alpha$.

Thus $\delta R_{\alpha\beta\gamma}\varepsilon$ is true if $\delta = \beta$ and $\varepsilon = \gamma$, but $\delta \bar{R}_{\alpha\beta\gamma}\varepsilon$ if either $\delta = \alpha$ or $\varepsilon = \alpha$.

The system ${}^2R_2^{(0)}$ we term the *projective analogue* of the system 1R_1 . It is true that the projective analogues ${}^3R_3^{(0)}$, ${}^4R_4^{(0)}$, etc., of our systems \S 2R_2 , 3R_3 , etc., respectively possess a genesis similar \parallel to that of ${}^2R_2^{(0)}$; but these analogues we shall discuss elsewhere \P .

It may be observed that the relation of the system ${}^1K_1^{**}$ to its corresponding projective analogue ${}^2K_2^{(0)}$ seems less simple than the relation of 1R_1 to ${}^2R_2^{(0)}$.

* In deriving these axioms we have augmented, for the purposes of the present paper, the conclusion of axioms 6 and 7 of 1R_1 by the statements " $\alpha R \xi_1$ " and " $\xi_2 R \beta$ " respectively. It may be noted that this change in the axioms does not invalidate the independence system given for axiom 4 of 1R_1 (T. G. R., p. 379).

\dagger The system ${}^2R_2^{(0)}$ does not, of course, justify the selection of a particular point; strictly, the linear descriptive system defined is a function of point variables.

\ddagger The definition of a linear descriptive system on the basis of linear projective order is, in itself, well known; cf. Russell, *loc. cit.*, § 204.

\S T. G. R., pp. 382, 387, etc.

\parallel Thus the projective analogue ${}^3R_3^{(0)}$ of 2R_2 arises by adjoining to the latter a point δ and assuming that $\alpha R \beta \gamma \delta$ implies $\gamma R \delta \alpha \beta$.

\P Compare T. G. R., p. 366.

** T. G. R., p. 394.

§ 3. *Relation to a Planar Descriptive System.*

Tactically, the linear projective axioms ${}^2R_2^{(0)}$ and the planar descriptive system ${}^2R_2^*$ are of the same dimensions: they are both stated in terms of (alternating) triads. A further comparison of the two theories discloses a remarkable resemblance.† For this purpose it is convenient to phrase the system 2R_2 as follows:‡

System 2R_2 .

1. There exists at least one point.
2. The existence of α implies the existence of $\alpha_0, \beta_0, \gamma_0$ such that $\alpha_0 R \beta_0 \gamma_0$ or $\alpha_0 R \gamma_0 \beta_0$.
3. $\alpha R \beta \gamma$ implies $\alpha \bar{R} \gamma \beta$.
4. $\alpha R \beta \gamma$ implies $\beta R \gamma \alpha$.
5. $\alpha R \beta \gamma, \xi \neq \alpha, \beta, \gamma$ imply $\xi R \beta \gamma$ or $\alpha R \xi \gamma$ or $\alpha R \beta \xi$.
6. $\alpha R \beta \gamma, \xi R \beta \gamma, \xi \neq \alpha$ imply $\alpha R \xi \gamma$ or $\xi R \alpha \gamma$ or $\alpha R \beta \xi$ or $\beta R \alpha \xi$.
7. $\alpha R \beta \gamma, \alpha \neq \beta, \beta \neq \gamma, \alpha \neq \gamma$ imply the existence of ξ such that $\beta R \xi \gamma$ and $\alpha R \gamma \xi$.
8. $\alpha R \beta \gamma, \varepsilon R \beta \gamma, \alpha R \varepsilon \gamma, \alpha R \beta \varepsilon, \varepsilon \neq \alpha, \beta, \gamma$ imply: the existence of ξ such that $\xi \bar{R} \alpha \varepsilon, \xi \bar{R} \varepsilon \alpha$, and the existence of δ' such that $\delta' R \beta \gamma$ implies $\delta' R \xi \gamma$ and $\delta' R \beta \xi$, and the existence of δ'' such that $\delta'' R \gamma \beta$ implies $\delta'' R \xi \beta$ and $\delta'' R \gamma \xi$.

Thus our linear axioms 1–4 are identical with axioms 1–4 of 2R_2 ; axioms 5 and 6 of ${}^2R_2^{(0)}$ affirm the truth of axioms 5 and 6 of 2R_2 . The resemblance between linear axiom 7 and the corresponding axiom of 2R_2 is evident. There is no linear projective axiom corresponding to axiom 8 of 2R_2 ; yet the latter axiom is consistent with our linear projective axioms, since the hypothesis of axiom 8 of 2R_2 is not fulfilled; compare theorems, § 1 of the present paper.

The preceding considerations show that the axioms of § 1 and the system 2R_2 can be logically combined into a single system merely by replacing, in axiom 7 of 2R_2 , the term $\alpha R \gamma \xi$ by $(\alpha R \xi \gamma \text{ or } \alpha R \gamma \xi)$. We have thus arrived

* T. G. R., p. 382.

† The essential basis for the resemblance in question is assumed to be identity (compare J. S. Mill, *loc. cit.*, Book I, Chapter IV, § 6, paragraphs 2, 3; also Bradley, "Appearance and Reality," London (1902), p. 592, III). This resemblance suggests that the geometry of the projective line may be considered as a quasi-planar descriptive geometry; in fact it seems not inappropriate to call the former a *singular* geometry of the descriptive plane.

‡ The present statement of our system 2R_2 does not differ essentially from that given in T. G. R., p. 382.

at a result which we believe to be a new application of the heuristic principle mentioned in our introduction.* With reference to this principle our result may be stated as follows :

The same general † theory underlies the theories of linear projective order and planar descriptive order. ‡

Further details of this general theory seem desirable.

§ 4. *Consistence and Independence of the Axioms of the General Theory.*

There exists at least one set of points which satisfies the axioms of the general theory of § 3, but which contravenes the systems ${}^2R_2^{(0)}$ and 2R_2 . As this set we may take the points in the interior and on the boundary of the unit circle in a cartesian plane and, say, continuous values of the coordinates; the relation $\alpha R \beta \gamma$ is valid if, and only if,

$$\begin{vmatrix} \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 1 \\ \gamma_1 & \gamma_2 & 1 \end{vmatrix} > 0 \quad \begin{cases} \alpha_1^2 + \alpha_2^2 \leq 1. \\ \beta_1^2 + \beta_2^2 \leq 1. \\ \gamma_1^2 + \gamma_2^2 \leq 1. \end{cases}$$

This region has the property that on its boundary the system ${}^2R_2^{(0)}$ is valid, while the system 2R_2 is satisfied in its interior.

We consider now the independence of the axioms of the general theory. As usual, we denote by C_i a class of points such that in this class axiom i ($i = 1, 2, \dots, 8$) is contradicted and the remaining axioms are satisfied or are non-effective.

C_8 . Construct a complete set of 35 ternary alternating products in seven elements, 1, 2, 3, 4, 5, 6, 7, such that every triad is reducible to the form $[\xi\eta\zeta]$, where $\xi < \eta < \zeta$. If in this set the triads

$$[125] \ [236] \ [347] \ [451] \ [562] \ [673] \ [714]$$

are replaced by their conjugates,

$$[215] \ [326] \ [437] \ [541] \ [652] \ [763] \ [174],$$

then the resulting set is a C_8 . Another set of the latter kind is obtained by considering a complete set of 84 ternary alternating products in nine elements,

* Cf. E. H. Moore, "Atti del IV Congresso Internazionale dei Matematici," Rome (1909), p. 98; O. Bolza, *Bulletin of the American Mathematical Society*, Vol. XVI (1910), p. 402.

† The generalization which is relevant to our application of the above principle is a *denotative* generalization, or generalization in extension, as distinguished from the *connotative* or intensive generalization.

‡ With this result compare Russell, *loc. cit.*, § 382.

1, 2, 3, . . . , 9, such that every triad is reducible to the form $[\xi\eta\zeta]$, where $\xi < \eta < \zeta$, and replacing the triads

[126] [237] [348] [459] [561] [672] [783] [894] [915]

by their conjugates. It would seem of interest to examine the availability of the infinite sequence of analogous sets that here suggests itself.

C_7 . Take any one of the complete sets of nC_3 ternary alternating products in n elements, 1, 2, 3, 4, . . . , n ($n = 3, 4, 5, \dots$), such that each product is reducible to the form $[\xi\eta\zeta]$, where $\xi < \eta < \zeta$.

C_6-C_1 . For the remaining independence sets reference may be made to AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909), p. 386.